Conditions for the occurrence of strong curvature singularities*

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Abstract. *Necessary and sufficient conditions for the occurrence of strong curvature singularities as defined by Tipler and Królak are derived. It is shown that both necessary and sufficient conditions involve not only the divergence of tetrad components of the Riemann, Ricci or Weyl tensor but also the divergence of their integrals along nonspacelike geodesics running into the singularity.*

1. INTRODUCTION

Let M be a space-time, by which we mean a smooth manifold M having dimension $n = 4$ equipped with a smooth metric g of Lorentzian signature $(-, +, +, +)$.

For a timelike geodesic $\gamma : [0, v_s) \to M$ e set $J_a(\gamma)$ (where $a \in [0, v_s)$) to be the set of maps $\mathbf{Z} : [a, v_s) \ni v \mapsto \mathbf{Z}(v) \in T_{\gamma(v)}(M)$ for which

$$(1) \qquad \frac{D^2 \mathbf{Z}}{dv^2} = R(\mathbf{K}, \mathbf{Z})\mathbf{K}$$

(where $\mathbf{K}$ is the tangent vector to γ) and $\mathbf{Z}(a) = 0$, $D\mathbf{Z}/dv\,|_a \in H_{\gamma(a)}$ where $H_{\gamma(a)}$ is the subspace of $T_{\gamma(a)}$ consisting of vectors orthogonal to $\mathbf{K}$. For a null geodesic

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$\gamma : [0, v_s) \to M$ we set $\hat{J}_a(\gamma)$ to be the set of maps $\hat{\mathbf{Z}} : [a, v_s) \ni v \mapsto \hat{\mathbf{Z}}(v) \in T_{\gamma(a)}(M)$ for which equation (1) holds with $\hat{\mathbf{Z}}$ substituted for $\mathbf{Z}$, $\hat{\mathbf{Z}}(a) = 0$ and $\hat{\mathbf{Z}}(a) \in S_{\gamma(a)}$, where $S_{\gamma(a)}$ is the quotient of $H_{\gamma(a)}$ consisting of equivalence classes of vectors in $H_{\gamma(a)}$ which differ only by a multiple of $\mathbf{K}(a)$.

Maps $\mathbf{Z}$, $\hat{\mathbf{Z}}$ are called Jacobi fields and (1) is Jacobi's equation. The correspondences $J_a(\gamma) \ni \mathbf{Z} \mapsto D\mathbf{Z}/\mathrm{d}v|_a \in H_{\gamma(a)}$ and $\hat{J}_a(\gamma) \ni \hat{\mathbf{Z}} \mapsto D\mathbf{Z}/\mathrm{d}v|_a \in S_{\gamma(a)}$ are linear isomorphisms.

If γ is timelike we assume the parametrization is by proper time and we set $\underset{1}{\mathbf{E}}, \underset{2}{\mathbf{E}}, \underset{3}{\mathbf{E}}, \underset{4}{\mathbf{E}} = \mathbf{K}$ to be an orthonormal tetrad parallely propagated on γ. If γ is null we assume it is affinely parametrized and the tetrad is chosen pseudo-orthonormal (i.e. $\underset{1}{\mathbf{E}}, \underset{2}{\mathbf{E}}$, spacelike and orthonormal and $\underset{3}{\mathbf{E}}, \underset{4}{\mathbf{E}}$ null with $g(\underset{3}{\mathbf{E}}, \underset{4}{\mathbf{E}}) = -1$), again with $\underset{4}{\mathbf{E}} = \mathbf{K}$.

For a timelike geodesic γ three linearly independent elements of $J_a(\gamma)$ define a volume element $\mu(v) = \underset{1}{\mathbf{Z}}(v) \wedge \underset{2}{\mathbf{Z}}(v) \wedge \underset{3}{\mathbf{Z}}(v) \in \Lambda^3(H_{\gamma(v)})$ at each point $\gamma(v)$ of γ. The map $\Delta : \Lambda^3(H_{\gamma(v)}) \to \mathbb{R}$ defined by $\Delta(\mathbf{A} \wedge \mathbf{B} \wedge \mathbf{C}) = \det[A^i, B^j, C^k]$ $(i, j, k = 1 \ldots 3)$ is independent of the choice of an orthonormal basis with $\underset{4}{\mathbf{E}} = \mathbf{K}$. We shall denote $\Delta(\mu(v))$ by $V(v)$. Similarly if γ is null two linearly independent elements $\underset{1}{\hat{\mathbf{Z}}}, \underset{2}{\hat{\mathbf{Z}}}$ of $J_a(\gamma)$ define an area element $\nu(v) = \underset{1}{\hat{\mathbf{Z}}}(v) \wedge \underset{2}{\hat{\mathbf{Z}}}(v) \in \Lambda^2(S_{\gamma(v)})$, and the map $\delta : \Lambda^2(S_{\gamma(v)}) \to \mathbb{R}$ defined by $\delta(B \wedge C) = \det[B^i, C^j]$ is independent of the choice of the pseudo-orthonormal basis $(i, j = 1, 2)$. We set $\delta(\nu(v)) = \hat{V}(v)$.

We consider the following two conditions on a timelike (respectively null) geodesic γ, which have been used by different authors to define strong curvature singularities.

Condition (T): For all $a \in [0, v_s)$ and any $\underset{1}{\mathbf{Z}}, \underset{2}{\mathbf{Z}}, \underset{3}{\mathbf{Z}} \in J_a(\gamma)$ (respectively $\underset{1}{\hat{\mathbf{Z}}}$, $\underset{2}{\hat{\mathbf{Z}}} \in \hat{J}_a(\gamma)$) we have that $\liminf_{v \to v_s} |V(v)| = 0$ (respectively $\liminf_{v \to v_s} |\hat{V}(v)| = 0$).

Condition (K): For all $a \in [0, v_s)$ and any $\underset{1}{\mathbf{Z}}, \underset{2}{\mathbf{Z}}, \underset{3}{\mathbf{Z}} \in J_a(\gamma)$ (respectively $\underset{1}{\mathbf{Z}}$, $\underset{2}{\mathbf{Z}} \in \hat{J}_a(\gamma)$) we have that there exists $v_1 \in (a, v_1)$ with $\mathrm{d}|\hat{V}(v)|/\mathrm{d}v|_{v_1} < 0$ (respectively $\mathrm{d}|V(v)|/\mathrm{d}v|_{v_1} < 0$).

The idea of a strong curvature singularity was introduced by Ellis and Schmidt [1]. A strong curvature singularity was distinguished by the property that all objects falling into it are crushed to zero volume, no matter what the physical properties of the object are. The concept of a strong curvature singularity has been defined in precise mathematical terms by Tipler [2]. Condition (T) above corresponds to his original definition; but it is equivalent to the published definitions where «lim inf» is replaced by «lim» because the length scale x (resp. $\hat{x}$) for which $V = x^3$ (resp. $\hat{V} = \hat{x}^2$) satisfies a propagation equation, discussed below, that makes it a concave function of the affine parameter, provided that the null convergence condition holds. It was conjectured by Tipler et al. [3] and independently by Królak [4] that all singularities in physically realistic space-times should be of strong curvature type.

The significance of this conjecture lies in a theorem proved by Królak which says, roughly, that if the conjecture is true and several regularity conditions are fulfilled then the weak version of Penrose's cosmic censorship hypothesis holds [5, 6]. It was realised by Królak that for the proof of this theorem a weaker restriction on the convergence of geodesics than that implied by Tipler's definition is sufficient. This led to the definition of a strong curvature singularity given by condition (K).

It was shown by Tipler [2] that there is a lower bound to the rate of the Ricci tensor growth along null geodesics satisfying condition (T) in a conformally flat space-time. In section 2 of this paper we shall set necessary conditions for the growth of Riemann curvature along both timelike and null geodesics satisfying conditions (T) and (K). We were able to improve the lower bound on the growth of Ricci curvature set by Tipler. In Section 3 we give sufficient conditions both for timelike and null cases for the growth of Ricci curvature in order that conditions (T) and (K) should be fulfilled. We also give sufficient conditions, in the null case only, for the growth of the Weyl tensor. In section 4 we express the necessary and sufficient conditions in terms of a comparison with power law behaviour.

The general conclusion is that the occurrence of strong curvature singularities implies the divergence not only of various tetrad components of the Riemann, Ricci and Weyl tensors but also of their integrals along geodesics approaching the singularity. Also certain integrals of components of curvature tensors must tend to infinity in order that strong curvature singularities may arise.

2. NECESSARY CONDITIONS FOR THE OCCURENCE OF STRONG CURVATURE SINGULARITIES

2.1. General results

We first prove two lemmas which constitute the basis for the proofs of all the necessary conditions.

LEMMA 1. *Suppose* $f : [0, v_s) \to \mathbb{R}$ *is integrable and satisfies*

$$f \geqslant 0, \int_0^{v_s} dv' \int_0^{v'} dv'' f(v'') < \infty. \tag{2}$$

Then for all $\eta > 0$, there exists $c(\eta) \in [0, v_s)$ such that for all a, v with $c \leqslant a \leqslant v$ we have

$$Q(a, v) = \int_a^{v_s} \mathrm{d}v' \int_a^{v'} \mathrm{d}v'' \, f(v'') < \infty. \qquad \blacksquare$$

Proof. By integration by parts

$$Q(a, v) = \int_a^{v} \mathrm{d}v' \, (v - v') \, f(v)$$

Hence we conclude that

$$Q(0, v) - Q(0, a) = \int_0^{a} \mathrm{d}v' (v - a) \, f(v) - Q(a, v)$$

i.e.

$$Q(a, v) \leqslant Q(0, v) - Q(0, a) \leqslant \int_a^{v} \mathrm{d}v' \left| \int_0^{v'} \mathrm{d}v'' \, f(v'') \right|.$$

But, by Cauchy's principle of convergence, the convergence in (2) implies that, for c sufficiently close to v_s, and all $a > c$, this integral is less than η, thus proving the lemma. ■

LEMMA 2. *Let $\mathbf{A} : [0, v_s) \to M_k$, $\mathbf{B} : [0, v) \to M_l$ be functions integrable on compact sets, where M_r is the set of real $r \times r$ matrices. Let $\mathbf{P} : M_l \to M_k$ be continuous and homogeneous of degree λ, and let $q : \mathbb{R}^k \to \mathbb{R}$ be a positive-definite quadratic form. Suppose that*

$$\int_0^{v_s} \mathrm{d}v' \int_0^{v'} \mathrm{d}v'' \, \| \mathbf{A}(v'') \| < \infty \tag{3}$$

$$\int_0^{v_s} \mathrm{d}v' \int_0^{v'} \mathrm{d}v'' \, \left\| \mathbf{P}\left(\int_0^{v''} \mathrm{d}v''' \, \| \mathbf{B}(v''') \| \right) \right\| < \infty \tag{4}$$

(where $\| \;\; \|$ denotes the mapping norm for matrices and the Euclidean norm for

vectors).

Then there exists an $a \in [0, v_s)$ *such that, for any vector* $\mathbf{Y}_a' \in \mathbb{R}^k$ *with* $\| Y_a' \| = 1$, *the solution* $\mathbf{Y}$ *to the system*

$$\frac{d^2 \mathbf{Y}(v)}{dv^2} = \mathbf{A}(v)\,\mathbf{Y}(v) + \mathbf{P}(\Sigma)\mathbf{Y} \tag{5}$$

$$\frac{d}{dv}\, q(\mathbf{Y}(v))\,\Sigma(v) = q(\mathbf{Y}(v))\,\mathbf{B} \tag{6}$$

with initial conditions

$$d\mathbf{Y}/dv|_a = \mathbf{Y}_a' \tag{7}$$

$$\mathbf{Y}(a) = 0 \tag{8}$$

(where $\mathbf{Y}$, Σ *are continuous on any closed interval* $[a, v]$ *of* $[0, v_s)$*) satisfies*

$$\| \mathbf{Y}(v) - (v - a)\,\mathbf{Y}_a' \| \leqslant 1/2\,(v - a) \tag{9}$$

for all v *in* $[a, v_s)$. ∎

Proof. We choose a by applying Lemma (1) to (3) and (4) so that

$$\int_a^v dv' \int_a^{v'} dv'' \, \| \mathbf{A}(v'') \| < \alpha \tag{10}$$

$$\int_a^v dv' \int_a^{v'} dv'' \, \left\| \mathbf{P}\left(\int_0^{v''} \mathbf{B}(v''')\, dv''' \right) \right\| < \beta \tag{11}$$

where α and β are constants to be fixed shortly.

From (7) we note that there exists $\delta > 0$ such that

$$\| \mathbf{Y}(a + \delta) - \delta\,\mathbf{Y}_a' \| \leqslant \delta/2$$

and so if we define

$$b = \sup\{ v : a < v < v_s \text{ and } \| \mathbf{Y}(v) - (v - a)\mathbf{Y}_a' \| \leqslant 1/2\,(v - a) \} \tag{12}$$

then we know that $b \geqslant a + \delta > a$.

We require to show that (9) holds for all $v \in [a, v_s)$, i.e. that $b = v_s$. To do this, we procede by contradiction: assume that $b < v_s$, in which case we should have that equality held in the condition in (12) at b (for if not then b could be increased). In other words

(13) $$\| \mathbf{Y}(b) - (b-a)\mathbf{Y}'_a \| = 1/2\,(b-a).$$

From (5)

(14) $$\Sigma(v) = \int_a^v \frac{q(\mathbf{Y}(v'))}{q(\mathbf{Y}(v))}\,\mathbf{B}(v')\,\mathrm{d}v'$$

and (12) gives

$$\frac{q(\mathbf{Y}(v'))}{q(\mathbf{Y}(v))} \leqslant \frac{9M}{m}$$

(from (12) with $v \leqslant b$) where

$$M = \sup_{\mathbf{x}} q(\mathbf{x})/\|\mathbf{x}\|^2$$

$$m = \inf_{\mathbf{x}} q(\mathbf{x})/\|\mathbf{x}\|^2.$$

Whence (14) becomes

$$\|\Sigma(v)\| \leqslant K_1 \int_a^v \|\mathbf{B}(v')\|\,\mathrm{d}v'$$

with $K_1 = 9M/m$.

Integrating (5) then gives, for $v < b$, using (12),

$$\left\| \frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}v}(v) - \mathbf{Y}'_a \right\| \leqslant \frac{3}{2}(v-a)\left[\int_a^v \|\mathbf{A}(v')\|\,\mathrm{d}v'\right] + $$

$$+ K_1{}^{\lambda} \int_a^v \left\|\mathbf{P}\left(\int_a^{v'} \|\mathbf{B}(v'')\|\,\mathrm{d}v''\right)\mathrm{d}v'\right.$$

and integrating again

$$\|\mathbf{Y}(v) - (v-a)\,\mathbf{Y}'_a\| \leqslant \frac{3}{2}\,(v-a)\,[\alpha - K_1{}^{\lambda}\beta].$$

Thus if we choose α and β so that

$$(\alpha + K_1^{\lambda}\beta) < \frac{1}{3}$$

then we obtain a contradiciton to (13), thus proving the Lemma. ■

2.2. Necessary conditions for the case (T)

We consider the timelike case explicitly, but an identical argument serves for the null case.

Suppose condition (T) holds. Consider an infinite sequence $\{v_i\}$ such that

$$\lim_{i \to \infty} v_i = v_s .$$

Let $\mathbf{A}_i = [\underset{1}{\mathbf{Z}}(v_i), \underset{2}{\mathbf{Z}}(v_i), \underset{3}{\mathbf{Z}}(v_i)]$ and $\epsilon_i = \det \mathbf{A}_i$. The characteristic polynomial $\det(\lambda I - A_i)$ must have a root of magnitude not greater than $|\epsilon_i|^{1/3}$; i.e. there exists a vector X_i (possibly complex) with $A_i X_i = \mu_i X_i$ such that $|\mu_i| \leqslant |\epsilon_i|^{1/3}$. Introducing a vector Y_i defined by $Y_i = X_i{}^1 \underset{1}{\mathbf{Z}} + X_i{}^2 \underset{2}{\mathbf{Z}} + X_i{}^3 \underset{3}{\mathbf{Z}}$ we have $\| Y_i \| = = \mu_i$. By condition (T) we can choose v_i so that $\epsilon_i \to 0$, and hence $\lim_{i \to \infty} \mu_i = 0$. So, if we now scale X_i so that $\| DY_i / \mathrm{d}v|_a \| = 1$ then we have that, for any in $[0, v_s)$ the following condition holds.

Condition $(T)'_a$: For all $\epsilon > 0$ and all $\mathrm{d} \in (a, v_s)$, there is a Jacobi field $\mathbf{Y} \in J_a(\gamma)$ with $\| D\mathbf{Y}/\mathrm{d}v|_a \| = 1$ and $\| \mathbf{Y}(v') \| < \epsilon$ for some $v' \in (\mathrm{d}, v_s)$ depending on ϵ and $\mathbf{Y}$.

Note that we have proved this for complex $\mathbf{Y}$. But the real and imaginary parts will separately satisfy the Jacobi equation and will have $\| \mathbf{Y}(v') \| < \epsilon$, so in condition $(T)_{a'}$, $\mathbf{Y}$ can be taken as real.

PROPOSITION 1. *For both timelike and null case, if condition (T) is satisfied then for some component $R^i{}_{4j4}$ of the Riemann tensor in a parallely propagated frame the integral*

$$I^i{}_j(v) = \int_0^v \mathrm{d}v' \int_0^{v'} \mathrm{d}v'' \, |R^i{}_{4j4}(v'')|$$

does not converge as $v \to v_s$. ■

Proof. We show the contrapositive form of the result: that if $I^i{}_j(v)$ does converge for every i, j then condition $(T)'_a$ is false for some a; i.e. that there exist a, d and ϵ such that $\| \mathbf{Y}(v) \| \geqslant \epsilon$ for all $v \in (d, v_s)$ and all Jacobi fields $\mathbf{Y} \in J_a(\gamma)$ with $\| \mathrm{d}\mathbf{Y}/\mathrm{d}v|_a \| = 1$.

We apply Lemma 2. We let the matrix $\mathbf{A}$ have components $|R^i{}_{4j4}|$ and we put $\mathbf{P}(\Sigma) = 0$. Equation (5) then becomes the Jacobi equation. Thus, if the integrals $I^i{}_j(v)$ converge, it follows from Lemma 2 that there is an a such that $\| \mathbf{Y} - (v - a)\mathbf{Y}'_a \| \leqslant 1/2\,(v - a)$ for all v in $[a, v_s)$. This implies that $\| \mathbf{Y} \| \geqslant$

$\geq 1/2\,(v-a) \geq 1/2\,(v_s - a) = \epsilon$ for $v > a + 1/2\,(v_s - a) = d$, say. In other words, $(T')_a$ is falsified with this choice of a, as required. ∎

One would also like to know the necessary conditions for strong curvature singularities to occur in terms of the Ricci and Weyl tensors since these have a well-known physical interpretation (see [7] chapter 4). To do so one must analyse the Jacobi equation in more detail.

Let $\mathbf{A}(v)$ be the matrix given by $\mathbf{A}(v) = [\underset{1}{\mathbf{Z}}(v), \underset{2}{\mathbf{Z}}(v), \underset{3}{\mathbf{Z}}(v)]$ where $\underset{1}{\mathbf{Z}}(v)$, $\underset{2}{\mathbf{Z}}(v)$, $\underset{3}{\mathbf{Z}}(v)$ are three linearly independent Jacobi fields in $J_a(\gamma)$. Let $\mathbf{K}$ be the matrix defined by $\mathbf{K} = \mathbf{A}^{-1}\mathrm{d}\mathbf{A}/\mathrm{d}v$. $\mathbf{K}$ can be expressed uniquely as $\mathbf{K} = \mathbf{W} + \Sigma + (1/3)\Theta\mathbf{I}$ where $\mathbf{W}$ is the antisymmetric part of $\mathbf{K}$ (called the vorticity matrix), Σ is the symmetric part of $\mathbf{K}$ (called the shear matrix) and Θ is the trace of $\mathbf{K}$ (called the expansion scalar). $\mathbf{I}$ is the unit matrix.

The equations for the propagation of $\mathbf{W}$, Σ and Θ are derived in chapter 4 of [7]. From the propagation equation for $\mathbf{W}$ it follows that if $\mathbf{W}$ vanishes initially then it must vanish everywhere wherever $\mathbf{A}$ is nonsingular (see [7] p. 87). This is the case for Jacobi fields in $J_a(\gamma)$.

The propagation equation for Θ, known as the Raychaudhuri equation, has the form

$$\frac{\mathrm{d}\Theta}{\mathrm{d}v} = -\frac{1}{3}\,\Theta^2 - 2\sigma^2 - R_{44}$$

where $2\sigma^2 = \mathrm{tr}(\Sigma^2)$. Since $V = \det \mathbf{A}$, by a well-known formula we have that

$$\Theta = \frac{1}{V}\,\frac{\mathrm{d}V}{\mathrm{d}v}\ .$$

Putting $x = V^3$ we can convert Raychaudhuri's euqation into the second order linear ordinary differential equation

$$\frac{\mathrm{d}^2 x}{\mathrm{d}v^2} = -\frac{1}{3}\,(2\sigma^2 + R_{44}), \tag{17}$$

A similar equation is obtained in the null case with 1/3 replace by 1/2 in equation (17). Note that x represents a length characteristic of the volume element V.

For the rest of the paper all quantities derived from $\mathbf{K}$ with a hat will refer to null geodesics.

As far as the shear tensor is concerned, we shall only need its propagation equation for the null case. It has the form

$$(18)\qquad \frac{d\hat{\Sigma}}{dv} = -\mathbf{C} - \hat{\Theta}\hat{\Sigma}$$

where $\mathbf{C}$ is a matrix with components $C^m{}_n = C^m{}_{4n4}$. Putting $\Theta = (2/x)(dx/dv)$ we can express equation (18) as

$$(19)\qquad \frac{d}{dv}\,\hat{x}^2\hat{\Sigma}^2 = -\hat{x}^2\mathbf{C}.$$

Since $V = x^3$ and $\hat{V} = \hat{x}^2$ we can give a condition for the occurrence of strong curvature singularities equivalent to condition (T) in terms of x and $\hat{x}$ as follows.

Condition $(T)'$: For all $a \in [0, v_s)$ and all solutions $x(v)$ of equation (17) (respectively solutions $\hat{x}(v)$) with the initial condition $x(a) = 0$ ($\hat{x}(a) = 0$) we have that $\liminf |x(v)| = 0$ (respectively $\liminf |\hat{x}(v)| = 0$).

From now on we shall assume that $R_{44} \geqslant 0$ both in the timelike and in the null cases. This assumption, called the timelike or null convergence condition is reasonable on physical grounds (see [7] article 4.3).

We shall be able to provide general necessary conditions for a strong curvature singularity to occur in terms of the Ricci and the Weyl tensor only for null geodesics.

PROPOSITION 2. *If $\gamma(v)$ is a null geodesic and condition $(T)'$ is satisfied then either the integral*

$$K(v) = \int_0^v dv' \int_0^{v'} dv''\, R_{44}(v'')$$

or the integral

$$L^m{}_n(v) = \int_0^v dv' \int_0^{v'} dv'' \left(\int_0^{v''} dv''' \,|\, C^m{}_{4n4}(v''') \,| \right)^2 .$$

for some m, n does not converge. ∎

Proof. The proof proceeds in exactly the same way as the proof of proposition 1. We first note that by putting $\mathbf{A}(v) = -(1/2)R_{44}(v)$, $\mathbf{B} = -\mathbf{C}$, $\mathbf{P}(\Sigma) = 1/2\, tr\, \Sigma^2$, $\mathbf{Y}(v) = \hat{x}(v)$ and $q(\hat{x}) = \hat{x}^2$ in equations (5) and (6) we obtain propagation equations for the characteristic length $\hat{x}$ and the shear matrix Σ described in this paragraph. We then show the contrapositive form of the result by applying Lemma 2. ∎

2.3. Necessary conditions for the case (K)

Again we consider only the timelike case explicitly. An identical argument holds for the null case. Suppose that condition (K) holds. We may suppose also that the stronger condition (T) does not hold, so that, for a sufficiently close to v_s, $\liminf_{v \to v_s} |V(v)| > 0$.

Writing as before $\mathrm{A}(v) = [\underset{1}{\mathbf{Z}}(v), \underset{2}{\mathbf{Z}}(v), \underset{3}{\mathbf{Z}}(v)]$ and $V(v) = \det \mathrm{A}(v)$, we have by a well-known formula, $\mathrm{d}V/\mathrm{d}v = V \operatorname{tr}(\mathrm{A}^{-1}\mathrm{d}\mathrm{A}/\mathrm{d}v)$. Thus condition (K) implies that for some v_0, $\operatorname{tr}(\mathrm{A}^{-1}\mathrm{d}\mathrm{A}/\mathrm{d}v)|_{v_0} < 0$. Let $\underset{i}{\mathbf{Y}} = \underset{j}{\mathbf{Z}}(\mathrm{A}(v_0)^{-1})^j{}_i$, so that $\underset{i}{\mathbf{Y}}(v_0)^j = \delta^j{}_i$.

Then

$$\operatorname{tr}\left(\mathrm{A}^{-1}\frac{\mathrm{d}\mathrm{A}}{\mathrm{d}v}\right)\Bigg|_{v_0} = \operatorname{tr}\frac{\mathrm{d}}{\mathrm{d}v}[\mathrm{A}(v)^{-1}\mathrm{A}]|_{v_0} =$$

$$= \operatorname{tr}\frac{\mathrm{d}}{\mathrm{d}v}[\underset{1}{\mathbf{Y}}(v), \underset{2}{\mathbf{Y}}(v), \underset{3}{\mathbf{Y}}(v)]|_{v_0} =$$

$$= \sum_i \frac{\mathrm{d}\underset{i}{\mathbf{Y}}^i}{\mathrm{d}v}\Bigg|_{v_0} = \sum_i g\left(\underset{i}{\mathbf{Y}}, \frac{\mathrm{d}\underset{i}{\mathbf{Y}}}{\mathrm{d}v}\right) = \frac{1}{2}\sum_i \frac{\mathrm{d}}{\mathrm{d}v} g(\underset{i}{\mathbf{Y}}, \underset{i}{\mathbf{Y}})|_{v_0}.$$

Thus condition (K) implies that for all a in $[0, v_s)$ the following condition holds:

Condition $(K')_a$: There is a Jacobi field $\mathbf{Y} \in J_a(\gamma)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}v} g(\mathbf{Y}, \mathbf{Y})\Bigg|_{v_0} < 0$$

for some $v_0 \in (a, v_s)$.

PROPOSITION 3. *For both the timelike and null cases, if condition (K) is satisfied then for some component $R^i{}_{4j4}$ of the Riemann tensor in a parallely propagated frame the integral*

$$J^i{}_j(v) = \int_0^v \mathrm{d}v' \, |R^i{}_{4j4}(v')|$$

does not converge as $v \to v_s$. ■

Again we prove the contrapositive by showing that, if all the $J^i{}_j$ converge, then an a can be found such that $(\mathrm{d}/\mathrm{d}v)g(\mathbf{Y}, \mathbf{Y}) > 0$ for all Jacobi fields $\mathbf{Y}$ in

$J_a(\gamma)$.

The convergence of $J^i{}_j(v)$ implies that the double integral $I^i{}_j(v)$ in Proposition 1 converges as well. Thus applying Lemma 2 in the same manner as in the proof of Proposition 1 we can ensure that there exists an a such that

$$\| \mathbf{Y}(v) - (v-a)\mathbf{Y}'_a \| \leqslant 1/2\,(v-a)$$

for all v in $[a, v_s)$.

Since $J^i{}_j(v)$ converge we can also choose a so that

$$\int_a^v | R^i{}_{4j4}(v') | \, \mathrm{d}v' < \frac{1}{6}$$

for all v in $[a, v_s)$. Then from the inequality (15) in the proof of Lemma 2 we have that

$$\| \frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}v} - \mathbf{Y}'_a \| \leqslant \frac{1}{4}\,(v-a).$$

Whence

$$\left| g\left(\mathbf{Y}, \frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}v}\right) - g(\mathbf{Y}'_a, (v-a)\mathbf{Y}'_a) \right| \leqslant | g(\mathbf{Y} - (v-a)\mathbf{Y}'_a, \mathbf{Y}'_a | +$$

$$+ \left| g\left(\mathbf{Y}, \frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}v} - \mathbf{Y}'_a\right) \right| \leqslant \frac{3}{4}\,(v-a).$$

Thus

$$g\left(\mathbf{Y}, \frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}v}\right) \geqslant g(\mathbf{Y}'_a, (v-a)\mathbf{Y}'_a) - \left| g\left(\mathbf{Y}, \frac{\mathrm{d}\mathbf{Y}}{\mathrm{d}v}\right) - g(\mathbf{Y}'_a, (v-a)\mathbf{Y}'_a) \right| \geqslant$$

$$\geqslant \frac{1}{4}\,(v-a) > 0.$$

This gives the required contradiction. ■

Again we shall provide necessary conditions in terms of the Ricci and the Weyl tensors for the null case. Using the characteristic length function $\hat{x}(v)$ condition (K) can be expressed as

Condition $(K)'$: For all $a \in [0, v_s)$ and all solutions $x(v)$ of equation (18) (respectively solutions $\hat{x}(v)$) with the initial condition $x(v) = 0$ ($\hat{x}(v) = 0$) we have that there exists $v_1 \in (a, v_s)$ with

$$\left.\frac{\mathrm{d}\,|x(v)|}{\mathrm{d}v}\right|_{v_1} < 0 \quad \left(\text{resp. } \left.\frac{\mathrm{d}\,|\hat{x}(v)|}{\mathrm{d}v}\right|_{v_1} < 0\right).$$

PROPOSITION 4. *If $\gamma(v)$ is a null geodesic and condition $(K)'$ is satisfied then either the integral*

$$M(v) = \int_0^v \mathrm{d}v' R_{44}(v')$$

or the integral

$$N^m{}_n(v) = \int_0^v \mathrm{d}v' \left(\int_0^{v''} \mathrm{d}v'' \, | C^m{}_{4n4}(v'') | \right)^2$$

for some m, n does not converge as $v \to v_s$. ∎

Proof. We again prove the contrapositive of the result. We assume that the integrals $M(v)$ and $N^m{}_n(v)$ converge. With the identifications made in the proof of Proposition 2 we have from Lemma 2 that there exists an a in $[0, v_s)$ such that

$$| \hat{x}(v) - (v - a)\hat{x}'_a | \leqslant (3/2)\,(v - a)$$

for all $v \in [a, v_s)$, where $\hat{x}'_a = \mathrm{d}\hat{x}/\mathrm{d}v\,|_a$. We choose $\hat{x}'_a = 1/2\,(v_s - a)$. It also follows that inequality (15) in the proof of Lemma 2 holds for all $v \in [a, v_s)$.

Thus since the integrals $M(v)$ and $N^m{}_n(v)$ converge, choosing a sufficiently close to v_s we have that

$$| \mathrm{d}\hat{x}/\mathrm{d}v - \hat{x}'_a | \leqslant 1/2\,(v - a)$$

for all v in $[a, v_s)$. Hence

$$\mathrm{d}\hat{x}/\mathrm{d}v \geqslant \hat{x}'_a - | \mathrm{d}\hat{x}/\mathrm{d}v - \hat{x}'_a | \geqslant \hat{x}'_a - 1/2\,(v - a) \geqslant 1/2\,(v_s - a) > 0.$$

Thus we have shown that there exists an a and a solution of equation (17) (with $1/2$ substituted for $1/3$ on the right hand side) with the initial condition $\hat{x}(a) = 0$ such that $\mathrm{d}\,|\hat{x}|/\mathrm{d}v > 0$ for all $v \in [a, v_s)$. This contradicts condition $(K)'$. ∎

Comment. The necessary conditions derived in propositions 2 and 4 are stronger than the conditions given in Propositions 1 and 3. It is possible to have a curvature tensor for which the integrals in Propositions 1 and 3 diverge, but the integrals in Propositions 2 and 4 converge. For example, take $R_{44} = 0$ and

$$R^m{}_{4n4} = C^m{}_{4n4} = [(v_s - v)\log(v_s - v)]^{-1} \quad (m, n = 1, 2).$$

It can be shown that the divergence of the integrals in Proposition 1 can be deduced from the divergence of those in Proposition 2, and the divergence of the integrals in Proposition 3 can be deduced from the divergence of those in Proposition 4.

3. SUFFICIENT CONDITIONS FOR THE OCCURRENCE OF STRONG CURVATURE SINGULARITIES

In order for the volume $V(v)$ (or $\hat{V}(v)$) to have inferior limit equal to zero it is not sufficient that one Jacobi fields should be focussed to zero, for the effect of this on the volume may be compensated by the other Jacobi fields becoming infinite. Therefore one must study the equations that determine the behaviour of $V(v)$ and $\hat{V}(v)$, i.e. the propagation equations for the characteristic lengths $x(v)$ and $\hat{x}(v)$ and the shear matrices Σ and Σ. Thus our sufficient conditions will be espressed in terms of the Ricci and the Weyl tensors.

PROPOSITION 5. *For both the timelike and the null cases, if the integral*

$$\int_0^v \mathrm{d}v' \int_0^{v'} \mathrm{d}v''\, R_{44}(v'')$$

diverges then condition (T) is satisfied. ■

Proof. It is clearly sufficient to prove, taking Jacobi fields in $J_a(\gamma)$ for any $a \in$ $\in [0, v_s)$, that for any $\epsilon > 0$, if (17) is satisfied with $x(a) = 0$, $x'(a) > 0$, then there exists $b \in (a, v_s)$ such that $x(b) < \epsilon$. Suppose the contrary. Then we have $x(b) > l$ for some l and all b in $[a, v_s)$. But then (17) becomes

$$\frac{\mathrm{d}^2 x}{\mathrm{d}v^2} < -\frac{l}{3} R_{44}$$

and the divergence of

$$\int_0^{v_s} \mathrm{d}v' \int_0^{v'} \mathrm{d}v''\, R_{44}(v'')$$

then contradicts $x(b) > l$. An identical argument holds for the null case. ■

An almost identical argument (using the fact that if initially $x'(a) > 0$, then there exist l and c with $a < c < v_s$ and $x(v) > l$ for $v > c$ and v less than the first maximum of x) shows that the following proposition holds.

PROPOSITION 6. *For both the timelike and null cases, if the integral*

$$\int_0^v \mathrm{d}v' \, R_{44}(v')$$

diverges to infinity, then condition (K) is satisfied. ■

A more difficult problem concerns singularities induced by the Weyl curvature $C^m{}_{4n4}$. Here we may have no reason to suppose that $C^m{}_{4n4} > 0$, and so the most natural condition physically would involve integrals of $|C^m{}_{4n4}|$. But it is possible for an integral of $|C^m{}_{4n4}|$ to diverge while $C^m{}_{4n4}$ oscillates in sign in such a way that no singularity is induced. This oscillation is analysed in the treatment of Szabados [8]. Here we consider, in the null case only, what happens if oscillation in excluded by supposing that, for some m and n, $C^m{}_{4n4}$ has a fixed sign.

PROPOSITION 7. *If $\gamma(v)$ is a null geodesic and, for some m, n we have $C^m{}_{4n4} > 0$ or $C^m{}_{4n4} < 0$ and the integral*

$$\int_0^v \mathrm{d}v' \int_0^{v'} \mathrm{d}v'' \left(\int_0^{v''} \mathrm{d}v''' \, |C^m{}_{4n4}(v''')| \right)^2$$

diverges, then condition (T) is satisfied. ■

Proof. We consider the case when $C^m{}_{4n4} < 0$. The proof for the case $C^m{}_{4n4} > 0$ is almost identical. As in the proof of Proposition 5 we suppose that condition (T) is not satisfied; that is there exists a solution $\hat{x}(v)$ of equation (17) with initial conditions $\hat{x}(a) = 0$ and $\hat{x}'(a) > 0$ such that $\hat{x}(b) > l$ for some l and all b in $[a, v_s)$.

Then from equation (19)

$$\frac{\mathrm{d}}{\mathrm{d}v} \hat{x}^2 \hat{\Sigma}^m{}_n > l^2 |C^m{}_{4n4}|.$$

Hence

$$\hat{x}^2 \hat{\Sigma}^m{}_n > l^2 \int_a^v \mathrm{d}v' \, |C^m{}_{4n4}(v')|.$$

Therefore from (17)

$$\frac{d^2\hat{x}}{dv^2} < -\frac{\hat{x}}{2}\left(\frac{l^2}{\hat{x}^2}\int_a^v dv' \,|\, C^m{}_{4n4}(v')\,|\right)^2 =$$

$$= -\frac{l^4}{2\hat{x}^3}\left(\int_a^v dv' \,|\, C^m{}_{4n4}(v')\,|\right)^2 \leqslant$$

$$\leqslant -\frac{l^4}{2(v_s - a)^3}\left(\int_a^v dv' \,|\, C^m{}_{4n4}(v')\,|\right)^2 .$$

Integrating the above inequality twice we obtain a contradiction with the supposition $\hat{x}(v) > l$. ■

In the same way as for proposition 4 we can show

PROPOSITION 8. *If $\gamma(v)$ is a null geodesic and, for some m, n, we have $C^m{}_{4n4} > 0$ or $C^m{}_{4n4} < 0$ and the integral*

$$\int_0^v dv'\left(\int_0^{v'} dv'' \,|\, C^m{}_{4n4}(v'')\,|\right)^2$$

diverges, then condition (K) is satisfied. ■

It is not possible to write down a condition that is both necessary and sufficient for case (T) or (K), even in the null case, without an analysis of the rotation of the eigenvectors of the Weyl tensor (i.e. of oscillations in its components). However analysing the propagation equations for expansion and shear in the null case we are able to obtain the same expressions for both necessary and sufficient conditions (compare Proposition 2 with Propositions 5 and 7 and Proposition 4 with Propositions 6 and 8).

4. COMPARISON WITH POWER LAW BEHAVIOUR

All the preceeding propositions yield corollaries in which conditions (T) and (K) are expressed in terms of a comparison with power law behaviour.

COROLLARY 1. *Let $\gamma(v)$ be either a timelike or a null geodesic. Let $R^i{}_{4j4}(v)$*

be a component of the Riemann tensor in a parallely propagated orthonormal or pseudo-orthonormal (if γ is null) frame. If, for all i, j and some fixed constant K, we have $|R^i{}_{4j4}| \leqslant K(v - v)^{-\alpha}$, then for $\alpha < 2$ condition (T) cannot be satisfied; while for $\alpha < 1$ condition (K) cannot hold. ■

Proof. The proof follows immediately from Propositions 1 and 2. ■

COROLLARY 2. *Let $\gamma(v)$ be a timelike geodesic and let R_{44} and $C^m{}_{4n4}$ be the components of the Ricci tensor and the Weyl tensor in a parallely propagated orthonormal frame. If, for some fixed constant K and some m, n we have $R_{44} \leqslant \leqslant K(v_s - v)^{-\alpha}$, $|C^m{}_{4n4}| \leqslant K(v_s - v)^{-\beta}$ with $\alpha = \beta < 2$, then condition (T) cannot be satisfied; while for $\alpha < 1$ and $\beta < 3/2$ condition (K) cannot hold.* ■

Proof. The proof follows from propositions 2 and 4. ■

COROLLARY 3. *Let the conditions of corollary 2 be satisfied. If for some fixed constant K either*

$$R_{44} \geqslant K(v_s - v)^{-\alpha}$$

or, for some m, n,

$$C^m{}_{4n4} \geqslant K(v_s - v)^{-\beta}$$

then for $\beta = \alpha \geqslant 2$ condition (T) bolds, while for $\beta \geqslant 3/2$, $\alpha \geqslant 1$ condition (K) holds. ■

Proof. The proof follows from Propositions 5, 6, 7 and 8. ■

5. CONCLUSION

In conclusion we stress that the occurrence of a singularity satisfying (*T*) does not in itself guarantee the existence of conjugate points on null geodesics. This is well-illustrated by the case

$$C^i{}_{4j4} = 0, \quad R_{44}(v) = M/(v - v_s)^2$$

on a timelike geodesic, where it is an elementary exercise to solve the differential equation for x and find that for $M \leqslant 3/4$ conjugate points do not occur, while for $M > 3/4$ they occur arbitrarily close to the singularity. But in both cases (by Corollary 3) condition (*T*) is satisfied.

Sufficient conditions for the occurrence of conjugate points in terms of the infimum of $|R^i{}_{4j4}|$ on bounded intervals have been derived by Newman [9].

Sufficient conditions for the occurrence of conjugate points along null geodesics in terms of integrals of ${C^m}_{4n4}$ have been derived by Szabados [8].

The sufficient conditions derived in section 3 and 4 provide a useful criterion for determining whether a given non-spacelike geodesic terminates in a strong curvature singularity. One possible application would be to determine whether any of the non-spacelike geodesic running into the naked singularity recently investigated by Christodoulou [10] satisfies conditions (T) or (K).

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